Critical phase in nonconserving zero-range processes and rewiring networks

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(Received 21 March 2005; published 25 October 2005)

Zero-range processes, in which particles hop between sites on a lattice, are closely related to rewiring networks, in which rewiring of links between nodes takes place. Both systems exhibit a condensation transition for appropriate choices of the dynamical rules. The transition results in a macroscopically occupied site for zero-range processes and a macroscopically connected node for networks. Criticality, characterized by a scale-free distribution, is obtained only at the transition point. This is in contrast with the widespread scale-free complex networks. Here we propose a generalization of these models whereby criticality is obtained throughout an entire phase, and the scale-free distribution does not depend on any fine-tuned parameter.

DOI: 10.1103/PhysRevE.72.046132 PACS number(s): 89.75.Hc, 05.70.Ln, 64.60.Ht

Many driven, nonequilibrium models reach a critical or scale invariant steady state only when their dynamical parameters are fine tuned to reach a phase transition point. Examples include a wide range of systems such as jamming in traffic [1], coalescence in granular gases [2], gelation in networks [3,4], and wealth condensation in macroeconomics [5]. In all these systems one has a condensation phase transition which we shall discuss in detail below. In other nonequilibrium models, for example in driven lattice gases and sandpile models [6,7], it has been argued that scale invariance and power-law distributions are generic, or at least one may have scale-free distributions across wide regions of the parameter space rather than just at critical points. This phenomenon has been termed self-organized criticality.

In recent years considerable attention has been given to the study of complex networks. Networks, defined as collections of nodes connected by links, are found in many fields of study, ranging from molecular biology to social communities and the Internet [8,9]. With each node one associates a degree k which is the number of links connected to it. In general links may be directed or they may carry a weight; however, for our purposes we do not consider such features. It has been observed that very often complex networks are characterized by a degree distribution $p(k)$ which decays algebraically for large k [8,9]. These networks, termed scale-free networks, are indeed critical, suggesting the existence of a mechanism which drives them to this state. Subsequently, dynamical processes for *growing* networks have been proposed in which nodes and links are continually added to the network with some predetermined rates [3,11]. The mechanism of linear preferential attachment, wherein new links attach to nodes with probability proportional to the degree of the nodes, results in a critical distribution for a wide range of the dynamical parameters [11]. On the other hand *rewiring* networks [9,12], whose dynamics constitutes rewiring processes with a fixed number of nodes, exhibit a critical distribution only at a critical point. This transition corresponds to condensation (also known as gelation) where a single node captures a finite fraction of the links.

Instructive insight has been gained into the condensation transition through the analysis of simple interacting particle systems [13–15]. These systems form fundamental models which may be mapped onto particular applications. For example, the zero-range process (ZRP) [16] is a particularly simple and exactly soluble model in which each site μ of a lattice contains an integer number of particles n_μ and particles hop to a neighboring site with rate $u(n_\mu)$. This model is closely related to the dynamics of rewiring networks [12].

Condensation occurs in the ZRP when the hopping rate $u(n)$ decays slowly enough with n, the number of particles at the site. The transition occurs on increasing the global conserved particle density $\rho=N/L$ where N is the number of particles and L is the number of sites. Then below a critical density ρ_c particles are thinly spread over all sites, but for $\rho>\rho_c$ a finite fraction of the particles condenses onto a single site. It has been shown (for a recent review see [13]) that condensation occurs if $u(n)$ decays asymptotically to some finite value β, as $u(n)\sim\beta(1+b/n)$ with $b>2$. In this case one finds that for $\rho<\rho_c$, the steady state probability that a site contains n particles, $p(n)$ decays exponentially and the system is in the low density, fluid phase. At the critical density, ρ_c, one has a power-law distribution $p(n)\sim n^{-b}$, thus a critical fluid. Above ρ_c, in addition to the power law, a piece of $p(n)$ emerges centered about $n=L(\rho-\rho_c)$; this piece represents the condensate [15] and contains the excess density. Thus the condensed phase corresponds to a critical fluid coexisting with a condensate. Only at criticality does one have a pure power-law distribution. The same condensation behaviour is seen in the degree distribution of rewiring networks with preferential attachment [9].

In this work we investigate how nonconservation affects the condensed phase of the ZRP. We show that the introduction of nonconservation in an appropriate fashion modifies the condensed phase into a scale-free phase by effectively suppressing the condensate and leaving the critical fluid. Thus a critical *phase* emerges as opposed to the conserving ZRP where criticality is only seen at a specific (conserved) density ρ_c. Using the relation to rewiring networks, through which the nonconservation corresponds to creation and annihilation of links, we show that the model provides a mechanism for generating power-law degree distributions. Thus, this mechanism plays a similar role in rewiring networks to that of linear preferential attachment in growing networks.

We begin by elucidating the mechanism for the generation of a critical phase within a generalized ZRP. Consider a lattice of L sites upon which reside a number of particles. With rate $u(n_\mu)$ (probability per unit time) which depends on the occupation n_μ of site μ, a particle is transferred from site μ to another site. We consider a fully connected geometry, where the destination site is chosen randomly from the other $L-1$ sites. In addition to the hopping dynamics, particles are added to site μ with a constant rate c, or removed with a rate $a(n_\mu)$, which increases with the site occupation n_μ. Thus, our choice for the dynamical rates is given by

$$n,m \to n-1,m+1 \quad \text{with rate } u(n) = \left(1 + \frac{b}{n}\right)\theta(n),$$

$$n \to n+1 \quad \text{with rate } c = \left(\frac{1}{L}\right)^s,$$

$$n \to n-1 \quad \text{with rate } a(n) = \left(\frac{n}{L}\right)^k, \tag{1}$$

where b, s, and k are positive parameters, and $\theta(n)$ is the usual Heaviside step function. Although the creation and annihilation might seem arbitrary, they in fact emerge quite naturally. The case $k=1$ corresponds to particles being annihilated independently, with order one annihilation events per unit time over the system. Including k as a parameter allows, for example, preferential annihilation at sites with large occupancy. The creation rate c does not depend on n but does depend on the system size L. This allows the overall creation rate in the system, which may be thought of as a driving rate, to be controlled. Also note that our results will hold for other choices of the rates which share the same asymptotic behavior (1).

The dynamical rates are conveniently implemented by using a random-sequential updating scheme, whereby at each time step a site μ is chosen at random, and a hop, annihilation or creation event may occur with relative probabilities given by the rates (1). To be explicit we consider the case $b>2$, although a similar analysis may be carried out for $b<2$ [10].

In Fig. 1 we compare the particle number distribution $p(n)$ of our model with that of the ZRP with conserving dynamics. It is clearly seen that for this choice of parameters, the creation-annihilation dynamics can selectively destroy any condensate and sustain the power-law distribution, corresponding to the critical fluid. In what follows we analyze the model showing that this feature holds for an entire region in the parameter space.

The steady state of the model is fully described by the probability distribution $P(n_1,n_2,\ldots,n_L)$ over all possible configurations. In contrast to the conserving ZRP [16], the steady state distribution of the model (1) does not factorize generally. However, we make the mean field approximation that the steady state distribution does factorize, i.e., $P(n_1,n_2,\ldots,n_L) \to \prod_{i=1}^{L} p(n_i)$. Due to the fully connected geometry we expect this approximation to become exact in the limit $L\to\infty$.

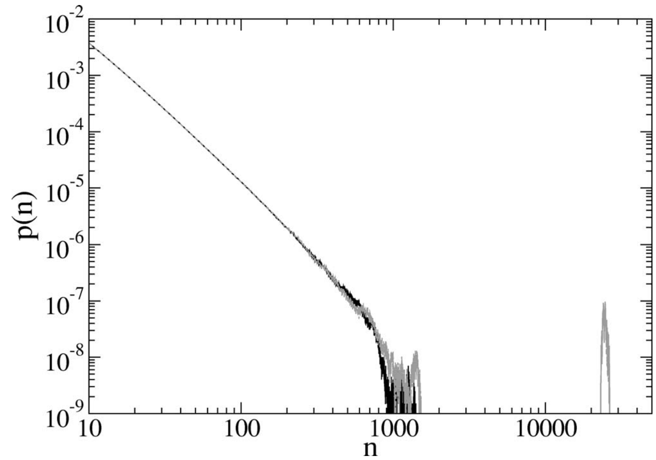

FIG. 1. Steady state distribution of the nonconserving ZRP (black solid line) and the conserving ZRP (gray dotted line). The data are from simulations run on a system of size $L=10^4$, with $b=2.6$, $k=3$, and $s=1.96$. In the conserving model the particle density was set to $\rho=4>\rho_c$. The peak at high occupation number, which exists only in the conserving model, corresponds to the condensate.

Using this approximation, the steady state master equation is given by

$$\begin{aligned}0 &= [u(n+1)+a(n+1)]p(n+1) - (\lambda + c)p(n)\\ &\quad - \{[u(n)+a(n)]p(n) - (\lambda+c)p(n-1)\}\theta(n).\end{aligned} \tag{2}$$

Here the current λ is given by

$$\lambda = \sum_{n=1}^{\infty} u(n)p(n). \tag{3}$$

From Eq. (2) it follows that

$$p(n) = \frac{(\lambda + c)^n}{\prod_{m=1}^{n}[a(m)+u(m)]}p(0). \tag{4}$$

Note that this is not a closed solution as λ depends on $p(n)$. The values of λ and $p(0)$ should be set such that both the normalization condition $1=\sum_{n=0}^{\infty}p(n)$ and the creation-annihilation balance condition

$$c = \sum_{n=1}^{\infty} a(n)p(n) \tag{5}$$

are obeyed. We now identify the three phases of the model by determining the asymptotic, large L behaviors of $p(n)$ and λ that satisfy Eqs. (3) and (5). The emergent phase diagram is summarized in Fig. 2. Deferring details to a later publication, we find the following results.

Low-density phase, $s>k$. Rewriting Eq. (5) as $L^{k-s} = \sum n^k p(n)$ implies $p(n)$ is a rapidly decreasing function of n, and the steady state density ρ is $\ll 1$. Thus, $p(1) \simeq \rho \sim L^{k-s}$ and in the thermodynamic limit the density goes to zero.

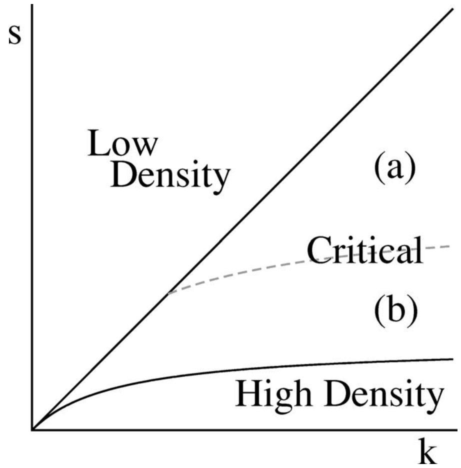

FIG. 2. Typical phase diagram of both ZRP and network models in the k-s plane with b fixed. k,s,b are defined in Eq. (1).

For $s<k$ the sum in Eq. (5) is controlled by the behavior of $p(n)$ at large n. We find the following regimes.

High-density phase, $s<k/(k+1)$. Here

$$p(n) \sim n^{-b}\exp\left(g_1\frac{n}{L^s} - \frac{n^{k+1}}{(k+1)L^k}\right), \tag{6}$$

where g_1 is a constant. Thus $p(n)$ is strongly peaked at $n \sim L^{1-s/k}$. This is the high density phase, where all sites are highly occupied. Note that the mean number of particles in the system, $N \sim L^{2-s/k}$ is superextensive.

Critical phase, $k>s>k/(k+1)$. In this phase the system relaxes to the critical density, and $p(n)$ takes an algebraic form. However, for large but finite systems two sub-phases are observed, distinguished by the finite-size corrections to the dominant power law.

For $k>s>kb/(k+1)$

$$p(n) \sim n^{-b}\exp\left(-g_2\frac{n}{L^x}\right), \tag{7}$$

where $x=(k-s)/(k-b+1)$ and g_2 is a constant. This cuts off the power law at $n \sim L^x$. We refer to this as critical subphase (a).

For $kb/(k+1)>s>k/(k+1)$

$$p(n) \sim n^{-b}\exp\left[n\left(\frac{d\ln L}{L}\right)^{k/(k+1)} - \frac{n^{k+1}}{(k+1)L^k}\right], \tag{8}$$

where $d=b-s(k+1)/k$. Here, on top of the algebraic part, $p(n)$ is weakly peaked at $n \sim L^{k/(k+1)}(\ln L)^{1/(k+1)}$. This peak will diminish as $L\to\infty$. We refer to this as critical subphase (b).

It is interesting to note that throughout critical subphase (a) the overall density $\rho=\int dn\, np(n)$ is given by the critical density corresponding to $p(n) \sim n^{-b}$. However in critical subphase (b), the density is controlled by a contribution from the weak peak. For $bk/(k+1)>s>2k/(k+1)$ the contribution of the weak peak to the density vanishes and the density is the critical density ρ_c as in subphase (a). However, for $2k/(k+1)>s>k/(k+1)$ the contribution from the weak peak to the density diverges as $L^{2k/(k+1)-s}$ making the number of particles in the system superextensive.

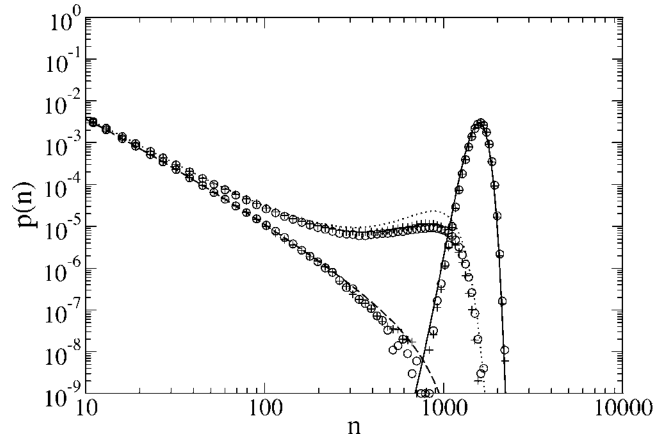

FIG. 3. Steady state distributions from simulations of the ZRP model on a fully connected lattice (+) and a 1D lattice (◯), compared with the theoretical asymptotic curves. Here $L=5000$ and $b=2.6$, $k=3$. Dashed curve is critical subphase (a) ($s=2$); dotted curve is critical subphase (b) ($s=1.2$); full line is high-density phase ($s=0.4$).

In Fig. 3 we present typical data obtained from numerical simulations in the different phases and compare with theoretical curves of $p(n)$. We found that in all phases, starting from random initial configurations of various densities, the system relaxes toward its expected steady state density. However, for the low-density phase the time scales for full relaxation were prohibitive and we do not present steady state data for this phase. In Fig. 3 we also provide data for a one-dimensional (1D) model, where sites are arranged in a 1D array, and particles are allowed to hop only to the right neighbor of the departure site. For the 1D geometry the mean-field approximation is not expected to be exact even in the limit $L\to\infty$. Nevertheless, we find numerically that the three phases discussed above exist also in the 1D model.

We now apply the approach discussed above to rewiring networks. To make the analogy with the ZRP, one identifies a site of the ZRP and its occupation number with a node in the network and its degree, respectively. We define a network model which incorporates both rewiring and creation-annihilation dynamics, and show how a proper choice of rates leads to the existence of a critical phase, much like that of the nonconserving ZRP, within which networks are scale-free. Due to the introduction of annihilation of links, there is no simple mapping from the ZRP to the network model since this would require keeping track of pairs of linked particles in the ZRP. However the two systems are closely related and as we shall see share the same phase diagram.

We consider a network of L nodes which are linked together by an integer number $N/2$ of undirected links (N is the number of particles in the corresponding ZRP). With rate $u(n_\mu)=1+b/n_\mu$ one of the n_μ links is disconnected from node μ and is rewired to another randomly chosen node. This does not change the number of links in the network. With rate $a(n_\mu)=(n_\mu/L)^k$ one of the links connected to node μ is removed from the network. In addition, a new link is created between node μ and another randomly chosen node at a constant rate $c=1/L^s$. For simplicity, here we allow multiple links between the same two nodes and self-connections to occur. Again the dynamics is conveniently implemented by choosing a node μ randomly at each time step, and changing the wiring with probabilities constructed from the relevant rates.

The mean-field master equation for the network model differs slightly from that of the ZRP (2), and is given by

$$0=[a(n+1)+u(n+1)+\Lambda(n+1)]p(n+1)-[a(n)+u(n)+\Lambda(n)]p(n)\theta(n)-[\lambda+2c]p(n)+[\lambda+2c]p(n-1)\theta(n). \tag{9}$$

Here $\lambda=\Sigma u(n)p(n)$ as before, and

$$\Lambda(n)=\frac{n}{N}\sum_{l=1}^{\infty}a(l)p(l)=c\frac{n}{N}. \tag{10}$$

The steady state solution to Eq. (9) is given by

$$p(n)=\frac{(\lambda+2c)^n}{\prod_{m=1}^{n}[\Lambda(m)+a(m)+u(m)]}p(0). \tag{11}$$

The main difference between (11) and the ZRP result (4) lies in $\Lambda(n)$ and its dependence on the total number of links in the system. This complicates the analysis slightly; however, all phases persist including the critical phase [10]. Thus the phase diagram in Fig. 2 also describes the network model: the low-density and high-density phases characterize extremely sparse and dense networks; the critical fluid corresponds to a scale-free phase in the thermodynamic limit. Examples of the degree distribution in the critical phase are given in Fig. 4, where we present data obtained from simulations of systems of increasing sizes. The power-law regime increases with system size.

Further interesting observations are that the weak peak of $p(n)$ in critical subphase (b) may correspond to a number of highly connected nodes but is distinct from a condensate. This would correspond to a large number of hubs in the network. Also, in the network model the suppression of events which link nodes to themselves or produce multiple links between the same two nodes may have a considerable effect on the results [17]. In the present model an additional cutoff is introduced into $p(n)$ and this can become the dominant scale. A full analysis will be published elsewhere [10].

Our main interest in the systems studied lies in the emergence of a critical phase which we have shown exists for annihilation and creation indices k,s in the range $k>s>k/(k+1)$. We conclude by comparing the critical phases we have identified to the critical points of the corresponding conserving ZRP and network models. In the latter, the average particle/link density ρ is an external parameter. A power-law distribution of the occupation number/degree is only obtained at $\rho=\rho_c$. In contrast, in the nonconserving models we have studied, a power-law distribution is obtained throughout the critical phase. In critical subphase (a) and in part of critical subphase (b) the steady state density is set by the dynamics to be ρ_c. However in the other part of critical subphase (b) the weak peak gives a diverging contribution to the density.

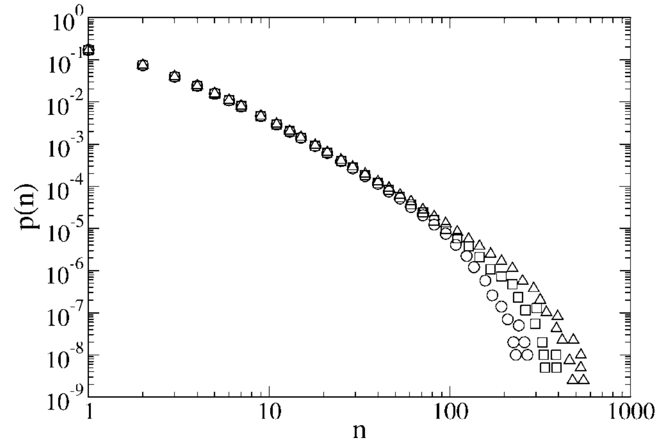

FIG. 4. Steady state probability distributions from simulations of the network model in the critical phase [subphase (a)]. Here $b=2.6$, $k=3$, and $s=2$, with $L=1000$ (circles), 2000 (squares), and 4000 (triangles).

In models exhibiting self-organized criticality, a critical phase is typically obtained only when the driving rate of the system vanishes with the system size [18,19], in order to ensure relaxation between stimuli. In comparison, in the present work the creation rate c in, e.g., Eq. (1), vanishes in the large L limit whereas u, the hopping or rewiring rate, does not vanish. Thus, there is a separation of time scales in the dynamical processes. On the other hand, there are no obvious avalanches or underlying absorbing states which are features usually associated with self organized criticality [20].

This study was supported by the Israel Science Foundation (ISF) and the Scottish Universities Physics Alliance (SUPA). Visits of M.R.E. to the Weizmann Institute were supported by the Albert Einstein Minerva Center for Theoretical Physics. Visits of D.M. to Edinburgh were supported by EPSRC Programme Grant No. GR/S10377/01.

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